Reduction Formula

1. $I_n = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx$ and $J_n = \int_0^1 \frac{x^{2n}}{(1+x^2)\sqrt{1-x^2}}dx$ (n = 0, 1, 2, 3, …)

(a) Prove that **(i)** $J_0 = \frac{\sqrt{2}}{2}I_0$ **(ii)** $2nI_n = (2n-1)I_{n-1}$.

(b) By considering $J_n + J_{n-1}$, show that the reduction formula for I_n allows J_n to be evaluated for any particular value of n .

(c) Prove that $J_3 = \frac{\pi(7-4\sqrt{2})}{16}$.

2. If $u_n = \int_0^{\pi/2} x\cos^n x dx$, where $n > 1$, show that $u_n = -\frac{1}{n^2} + \frac{n-1}{n}u_{n-2}$.

Evaluate u_4 and u_5 .

3. If $I_n = \int \frac{x^n}{\sqrt{a^2+x^2}}dx$, show that $I_n = \frac{x^{n-1}}{n}\sqrt{a^2+x^2} - \frac{n-1}{n}a^2 I_{n-2}$, where $n \geq 2$.

Evaluate $\int_0^2 \frac{x^5}{\sqrt{5+x^2}}dx$.

4. If $u_m = \int x^m(a^2-x^2)^{1/2}dx$, show that $(m+2)u_m = -x^{m-1}(a^2-x^2)^{3/2} + a^2(m-1)u_{m-2}$.

Evaluate $\int_0^{\pi/2} \sin^{2m}\theta\cos^2\theta d\theta$, where m is a positive integer.

5. If $u_n = \int x^n(2ax-x^2)^{1/2}dx$, where n is a positive integer, prove that:

$(n+2)u_n - (2n+1)au_{n-1} + x^{n-1}(2ax-x^2)^{3/2} = 0$.

Show that $\int_0^{2a} x^2(2ax-x^2)^{1/2}dx = \frac{5\pi a^4}{8}$.

6. If $I_n = \int_0^a \frac{x^n}{\sqrt{3a^2+x^2}}dx$, show that $I_n = \frac{2a^n}{n} - \frac{3(n-1)}{n}a^2 I_{n-2}$. Evaluate I_7 .

7. If $I_{p,q} = \int \frac{x^p}{(1+x^2)^q}dx$, show that $2(q-1)I_{p,q} = -\frac{x^{p-1}}{(1+x^2)^{q-1}} + (p-1)I_{p-2,q-1}$.

Hence, or otherwise, evaluate $\int_0^1 \frac{x^6}{(1+x^2)^3}dx$.

8. If $I_n = \int_0^{2a} x^n\sqrt{2ax-x^2}dx$, prove that $3aI_n - 3I_{n+1} = n(I_{n+1} - 2aI_n)$.

Hence, or otherwise, show that $I_n = \frac{(2n+1)(2n-1)...7\cdot 5}{(n+2)(n+1)...5\cdot 4} \times \frac{\pi}{2}a^{n+2}$, where n is a positive integer.

9. Let $I_n(z) = \int_0^1 (1-y)^n(e^{yz}-1)dy$, for all $n \geq 0$.

Prove that for all $n \geq 1$, $I_{n-1}(z) = \frac{z}{n}I_n(z) + \frac{z}{n(n+1)}$. Deduce that $e^z = \sum_{r=0}^{n}\frac{z^r}{r!} + \frac{z^n}{(n-1)!}I_{n-1}(z)$.

10. Find a reduction formula for the integral $I_n = \int_0^1 (1-x^2)^n dx$ and use it to evaluate $\int_0^1 (1-x^2)^8 dx$.

11. **(a)** Prove that $\int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$.

(b) If $I_n = \int_0^a x^n (a^2 - x^2)^{1/2} dx$, $n > 1$, prove that $I_n = \left(\frac{n-1}{n+2}\right) a^2 I_{n-2}$.

Hence find I_4 .

12. **(a)** Prove the ***Wallis formulae***:

(i) $\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{(2n)!}{2^{2n} n! n!} \frac{\pi}{2}$

(ii) $\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2^{2n} n! n!}{(2n+1)!}$

(iii) $\int_0^{\pi/2} \sin^{2n-1} x \, dx = \left(1 + \frac{1}{2n}\right) \int_0^{\pi/2} \sin^{2n+1} x dx$

(b) Prove that $0 < x < \frac{\pi}{2}$, then $\int_0^{\pi/2} \sin^{2n} x \, dx = \left(1 + \frac{\theta_n}{2n}\right) \int_0^{\pi/2} \sin^{2n+1} x dx$, where $0 < \theta_n < 1$.

(c) Deduce that $\int_0^{\pi/2} \sin^{2n} x \, dx = \sqrt{1 - \frac{1-\theta_n}{2n+1}} \sqrt{\frac{\pi}{n}} \cdot \frac{1}{2}$ and deduce that $\frac{(2n)!}{2^{2n} n! n!} = \sqrt{1 - \frac{1-\theta_n}{2n+1}} \frac{1}{\sqrt{n\pi}}$.

13. **(a)** Obtain a reduction formula for $\int_0^{\pi/4} \left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)^{2n+1} dx$ and hence deduce its value.

(b) Prove that $\int_0^{\pi/2} (\ln \cos x) \cos x dx = \ln 2 - 1$.

14. If $I(p,q) = \int_b^a (x-a)^p (b-x)^q dx$ $(b > a)$, prove that when $n \geq 1$,

(i) $I(n, n-1) = I(n-1, n)$

(ii) $2(2n+1) I(n, n) = 2n (b-a) I(n, n-1)$

$= n(b-a)^2 I(n-1, n-1)$.

Hence or otherwise, show that $I(n,n) = \frac{(b-a)^{2n+1} n! n!}{(2n+1)!}$.

15. Find a reduction formula for $\int (a^2 + x^2)^{n/2} dx$, where n is an odd positive integer.

Prove that $\int_0^2 (5 + x^2)^{3/2} dx = \frac{396 + 75 \ln 5}{16}$.